An explicit HHO method for the wave equation

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The Hybrid High-Order (HHO) method

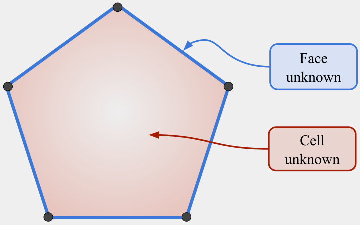
- Introduced in 2014 for the linear diffusion & elasticity (Di Pietro, Ern, Lemaire 2014; DP, E 2015)
- Extended to nonlinear mechanics, electromagnetism, Stokes, fluid mechanics...

Links with other methods

- Bridged to Hybridizable Discontinuous Galerkin HDG (Cockburn, Di Pietro, Ern 2016)
- and to Nonconforming Virtual elements ncVEM
- Same devising principle as weak Galerkin WG (Wang, Ye 2013) but with optimal stabilisation

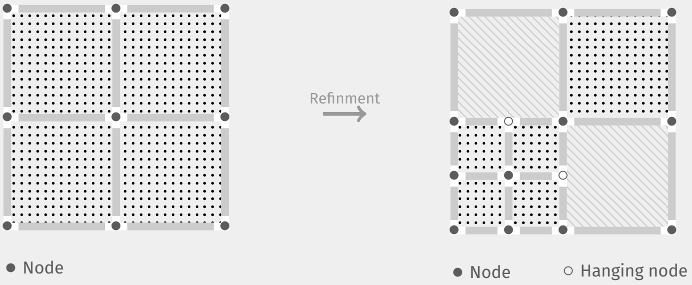
Characteristics of HHO

- Hybrid unknowns
  - **face and cells unknowns**
- Reconstructed gradient and stabilisation
- High order of convergence



Advantages of HHO

- HHO vs FE
  - Support of polyhedral meshes → natural use for adaptive mesh refinement
  - Locking-free method (elasticity)
  - Cell mass matrix is naturally block diagonal
- HHO vs DG
  - Nonlinear flux is manipulated on the cells only
  - Symmetric formulation for nonlinear problems
- High Order → efficient to counter dispersion



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DE LA RECHERCHE À L’INDUSTRIE

Diffusion problems

Problem

$$\begin{cases} -\nabla \cdot (\mu \nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Mesh

Let $\mathcal{C}$ be a polyhedral subdivision of domain Ω with

- cells $C \in \mathcal{C}$
- faces $F \in \mathcal{F}$

h_C : diameter of a cell C.

h : maximum of all cell diameters h_C.

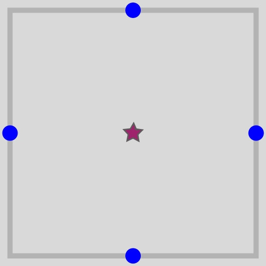
μ_C : diffusion coefficient, **cell-wise constant** for simplicity

Degrees of freedom

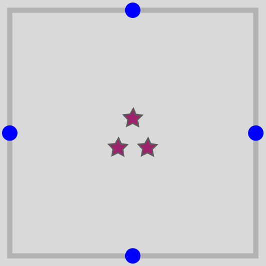
Discrete space

- **Faces** : $w_F \in P^k(F)$ and **Cells** : $w_C \in P^l(C)$ with $l \in \{k, k+1\}$
- **Dofs of a cell** : $\hat{w}_C = (w_C, (w_F)_{F \in \partial C}) \in \hat{\mathcal{U}}_C = P^l(C) \times \bigtimes_{F \subset \partial C} P^k(F)$
- **Dofs in Ω +BC's** : $\hat{w}_h \in \hat{\mathcal{U}}_{h,0} = \{((w_C)_{C \in \mathcal{C}}, (w_F)_{F \in \mathcal{F}}, w_F = 0 \text{ for } F \in \partial\Omega)\}$

Degrees of freedom



$k = 0, l = 0$



$k = 0, l = 1$

Goal : Reconstruct locally a gradient $G_C(\hat{u}_C)$ consistent with faces and cells DOFs

Definition

$G_C(\hat{u}_C) : \hat{\mathcal{U}}_C \to \nabla\mathcal{P}^{k+1}(C)$ defined for all $\hat{u}_C = (u_C, (u_F)_{F\in\partial C}) \in \hat{\mathcal{U}}_C$,

$$(G_C(\hat{u}_C), \nabla w)_c = (\nabla u_C, \nabla w)_c + (u_{\partial C} - u_{C|\partial C}, \nabla w_{|\partial C} \cdot \vec{n})_{\partial C}, \ \forall w \in \mathcal{P}^{k+1}(C)$$

mimics integration by parts in C.
One can also define a **potential reconstruction operator** $p_C(\hat{u}_C) \in \mathcal{P}^{k+1}(C)$ which verifies

$$\nabla p_C(\hat{u}_C) = G_C(\hat{u}_C), \quad \int_C p_C(\hat{u}_C) = \int_C u_C$$

Stabilisation operator

Goal : Enforce in a weak manner the following consistency condition

$$\delta(v) = v_{C|F} - v_F \simeq 0$$

Mixed-order (Lehrenfeld-Schöberl stabilisation for HDG)

$$s_C(\hat{v}_C, \hat{w}_C) := \sum_{F \in \partial C} \frac{\mu_C}{h_C} \left(\pi_F^k \delta(v), \pi_F^k \delta(w) \right)_F$$

Without π_F^k : plain least-squares stabilisation $\rightarrow$ suboptimal (often used in WG).

Equal-order

$$s_C(\hat{v}_C, \hat{w}_C) = \sum_{F \in \partial C} \frac{\mu_C}{h_C} \left(\pi_F^k(\delta(v) - (I - \pi_C^k) p_C(0, \delta(v))), \pi_F^k(\delta(w) - (I - \pi_C^k) p_C(0, \delta(w))) \right)_F$$

- First stabilisation using the reconstructed gradient in HDG context
- More costly than the mixed-order case

Discrete formulation

$$\hat{u}_h \in \hat{\mathcal{U}}_{h,0} \text{ such that } , \sum_{C\in\mathcal{C}} (\mu_C G_C(\hat{u}_C), G_C(\hat{w}_C))_C + s_C(\hat{u}_C, \hat{w}_C) = \sum_{C\in\mathcal{C}} (f, w_C)_C, \forall \hat{w}_h \in \hat{\mathcal{U}}_{h,0}$$

Algebraic formulation

$$\begin{bmatrix} \mathbb{A}_{\mathcal{C}\mathcal{C}} & \mathbb{A}_{\mathcal{C}\mathcal{F}} \\ \mathbb{A}_{\mathcal{F}\mathcal{C}} & \mathbb{A}_{\mathcal{F}\mathcal{F}} \end{bmatrix} \begin{bmatrix} U_{\mathcal{C}} \\ U_{\mathcal{F}} \end{bmatrix} = \begin{bmatrix} F_{\mathcal{C}} \\ 0 \end{bmatrix},$$

with $\mathbb{A} = \mathbb{K} + \mathbb{S}$ ($\mathbb{K}$ stiffness, $\mathbb{S}$ stabilisation)

Static condensation

- Using the block-diagonal structure of $\mathbb{A}_{\mathcal{C}\mathcal{C}}$

$$\left(\mathbb{A}_{\mathcal{F}\mathcal{F}} - \mathbb{A}_{\mathcal{F}\mathcal{C}}\mathbb{A}_{\mathcal{C}\mathcal{C}}^{-1}\mathbb{A}_{\mathcal{C}\mathcal{F}}\right) U_{\mathcal{F}} = \tilde{F}_{\mathcal{F}}$$

- Once $U_{\mathcal{F}}$ is computed, $U_{\mathcal{C}}$ is obtained by a local post-processing

Error analysis

Energy norm (Di Pietro, Ern, Lemaire 2014)

Let $u \in H_0^1(\Omega)$ and $\hat{u}_h \in \hat{\mathcal{U}}_{h,0}$ be the continuous and discrete solutions. Assume $u \in H^{k+2}(\mathcal{C})$. Then

$$\|\nabla u - G_{\mathcal{C}}(\hat{u}_h)\|_{L^2(\Omega)} \lesssim h^{k+1} \|u\|_{H^{k+2}(\mathcal{C})}.$$

with $G_{\mathcal{C}}(\hat{u}_h)_{|C} = G_C(\hat{u}_C) \quad \forall C \in \mathcal{C}$.

L^2 norm

Under the same assumptions, elliptic regularity and $f \in H^1(\Omega)$ if $k = 0$,

$$\left\| \pi_{\mathcal{C}}^l(u) - u_{\mathcal{C}} \right\|_{L^2(\Omega)} \lesssim h^{k+2} \|u\|_{H^{k+2}(\mathcal{C})}$$

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LINEAR WAVE EQUATION

$$\begin{cases} \boxed{\partial_t^2 u} - \nabla \cdot (\mu \nabla u) = f & \text{in } \Omega \times J \\ u_{|t=0},\ \partial_t u_{t=0} = u_0, v_0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \times J \end{cases}$$

with $f \in C^0(J; L^2(\Omega))$, $u_0, v_0 \in H_0^1(\Omega)$ and $J := [0; T_f]$.

Space semi-discrete problem

Find $\hat{u}_h(\cdot, t) \in C^2(J; \hat{\mathcal{U}}_{h,0})$ such that, for all $\hat{w}_h \in \hat{\mathcal{U}}_{h,0}$ and all $t \in J$

$$\boxed{(\partial_t^2 u_{\mathcal{C}}(\cdot, t), w_{\mathcal{C}})_\Omega} + \sum_{C \in \mathcal{C}} (\mu_C G_C(\hat{u}_C(\cdot, t), G_C(\hat{w}_C))_C + s_C(\hat{u}_C(\cdot, t), \hat{w}_C) = (f, w_{\mathcal{C}})_\Omega$$

Error analysis (Burman, Duran, Ern, Steins, 2020)

Let u be the continuous solution and $\hat{u}_h$ the space semi-discrete solution.

$$\|\partial_t(u - u_{\mathcal{C}})\|_{L^\infty(J;L^2)} + \|\nabla u - G_{\mathcal{C}}(\hat{u}_h)\|_{L^\infty(J;L^2)} \lesssim h^{k+1} \left(\|u\|_{L^\infty(J;H^{k+2})} + T\|\partial_t u\|_{L^\infty(J;H^{k+2})} \right)$$

Semi discrete equation

$$\begin{bmatrix} \mathbb{M}_{\mathcal{C}\mathcal{C}} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \partial_t^2 U_{\mathcal{C}}(t) \\ \cdot \end{pmatrix} + \begin{bmatrix} \mathbb{A}_{\mathcal{C}\mathcal{C}} & \mathbb{A}_{\mathcal{C}\mathcal{F}} \\ \mathbb{A}_{\mathcal{F}\mathcal{C}} & \mathbb{A}_{\mathcal{F}\mathcal{F}} \end{bmatrix} \begin{pmatrix} U_{\mathcal{C}}(t) \\ U_{\mathcal{F}}(t) \end{pmatrix} = \begin{bmatrix} F_{\mathcal{C}}(t) \\ 0 \end{bmatrix} \quad \forall\, t \in J.$$

Problem

The second equation induces a static coupling between cell and face unknowns

$$\mathbb{A}_{\mathcal{F}\mathcal{F}} U_{\mathcal{F}}(t) = -\mathbb{A}_{\mathcal{F}\mathcal{C}} U_{\mathcal{C}}(t)$$

- Possibility to write the problem as first-order problem and remove the static coupling in the mixed-order case (Burman, Duran, Ern 2022)
- In first-order formulation, stabilisation dissipates exact energy, whereas in second-order formulation a discrete energy in conserved.

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EXPLICIT TIME INTEGRATION

EXPLICIT TIME INTEGRATION

Leapfrog scheme

Discretize J in $N+1$ time nodes $t^0, t^1, \dots, t^N$.

$$\frac{1}{\Delta t^2}\begin{bmatrix} \mathbb{M}_{\mathcal{C}\mathcal{C}} & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} U_{\mathcal{C}}^{n+1} - 2U_{\mathcal{C}}^{n} + U_{\mathcal{C}}^{n-1} \\ \cdot \end{pmatrix} + \begin{bmatrix} \mathbb{A}_{\mathcal{C}\mathcal{C}} & \mathbb{A}_{\mathcal{C}\mathcal{F}} \\ \mathbb{A}_{\mathcal{F}\mathcal{C}} & \mathbb{A}_{\mathcal{F}\mathcal{F}} \end{bmatrix}\begin{pmatrix} U_{\mathcal{C}}^{n} \\ U_{\mathcal{F}}^{n} \end{pmatrix} = \begin{bmatrix} F_{\mathcal{C}}^{n} \\ 0 \end{bmatrix}$$

unknowns : $U_{\mathcal{C}}^{n+1}$ and $U_{\mathcal{F}}^{n}$.

This is solved in two steps :

- ✗ Solving the coupling problem between the cell and the face unknowns

$$\mathbb{A}_{\mathcal{F}\mathcal{F}} U_{\mathcal{F}}^{n} = -\mathbb{A}_{\mathcal{F}\mathcal{C}} U_{\mathcal{C}}^{n}$$

- ✓ The "evolution" equation on the cells unknowns at time t^{n+1} with the mass matrix $\mathbb{M}_{\mathcal{C}\mathcal{C}}$ which is block-diagonal

$$\mathbb{M}_{\mathcal{C}\mathcal{C}} U_{\mathcal{C}}^{n+1} = F_{\mathcal{C}}^{n} + \mathbb{M}_{\mathcal{C}}(2U_{\mathcal{C}}^{n} - U_{\mathcal{C}}^{n-1}) - \Delta t^2(\mathbb{A}_{\mathcal{C}\mathcal{C}} U_{\mathcal{C}}^{n} + \mathbb{A}_{\mathcal{C}\mathcal{F}} U_{\mathcal{F}}^{n})$$

Iterative splitting on the faces

Mixed-order

$$\mathbb{A}_{\mathcal{F}\mathcal{F}} U_{\mathcal{F}}^n = (\mathbb{K}_{\mathcal{F}\mathcal{F}} + \mathbb{S}_{\mathcal{F}\mathcal{F}}) U_{\mathcal{F}}^n = -\mathbb{A}_{\mathcal{F}\mathcal{C}} U_{\mathcal{C}}^n$$

is transformed into iterated problems ($\mathbb{S}_{\mathcal{F}\mathcal{F}}$ is block-diagonal)

$$\mathbb{S}_{\mathcal{F}\mathcal{F}} U_{\mathcal{F}}^{n-1,m} = -\mathbb{A}_{\mathcal{F}\mathcal{C}} U_{\mathcal{C}}^n - \mathbb{K}_{\mathcal{F}\mathcal{F}} U_{\mathcal{F}}^{n-1,m-1}$$

Equal-order

$$\mathbb{A}_{\mathcal{F}\mathcal{F}} U_{\mathcal{F}}^n = (\mathbb{K}_{\mathcal{F}\mathcal{F}} + \mathbb{S}^{\star}_{\mathcal{F}\mathcal{F}} + \mathbb{S}'_{\mathcal{F}\mathcal{F}}) U_{\mathcal{F}}^n = -\mathbb{A}_{\mathcal{F}\mathcal{C}} U_{\mathcal{C}}^n$$

is transformed into iterated problems

$$\mathbb{S}^{\star}_{\mathcal{F}\mathcal{F}} U_{\mathcal{F}}^{n-1,m} = -\mathbb{A}_{\mathcal{F}\mathcal{C}} U_{\mathcal{C}}^n - (\mathbb{K}_{\mathcal{F}\mathcal{F}} + \mathbb{S}'_{\mathcal{F}\mathcal{F}}) U_{\mathcal{F}}^{n-1,m-1}$$

with $\mathbb{S}_{\mathcal{F}\mathcal{F}} = \mathbb{S}^{\star}_{\mathcal{F}\mathcal{F}} + \mathbb{S}'_{\mathcal{F}\mathcal{F}}$ and $\mathbb{S}^{\star}_{\mathcal{F}\mathcal{F}}$ block-diagonal.

Analysis

This iterative splitting is equivalent to a Neumann series to invert $\mathbb{A}_{\mathcal{F}\mathcal{F}}$.

- **Condition on the spectral radius** $\mathbb{S}_{\mathcal{F}\mathcal{F}}^{-1} \mathbb{K}_{\mathcal{F}\mathcal{F}} < 1$ (or $\mathbb{S}_{\mathcal{F}\mathcal{F}}^{-\star} (\mathbb{K}_{\mathcal{F}\mathcal{F}} + \mathbb{S}'_{\mathcal{F}\mathcal{F}}) < 1$).
- This condition can be achieved by *scaling* $\mathbb{S}$ by β *large enough*.

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NUMERICAL RESULTS

Scaling of stabilisation

- β is lower bounded by a value $\beta^\star$ independent of h
  - $\beta^\star$ depends on **the trace constant** C_{tr} s.t. $\|v\|_F \leqslant C_{tr} h_C^{-1/2} \|v\|_C, \forall v \in \mathcal{P}^k(C)$
  - $\beta^\star$ depends on $\boldsymbol{k}$ and the **mesh regularity**
- In practice $\beta^\star$ is computed on a **coarse mesh with Neumann boundary conditions**
- Mild overestimation on reasonably fine meshes with Dirichlet conditions

- Analytical solution $u(x, y, t) = t^2 \sin(\pi x) \sin(\pi y)$
- $\Omega = [0; 1]^2$

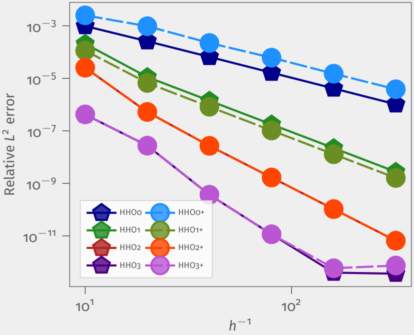

Figure. – L^2-error convergence curves : h^{k+2}

Scaling of stabilisation

- β is lower bounded by a value $\beta^\star$ independent of h
  - $\beta^\star$ depends on **the trace constant** C_{tr} s.t. $\|v\|_F \leqslant C_{tr} h_C^{-1/2} \|v\|_C, \forall v \in \mathcal{P}^k(C)$
  - $\beta^\star$ depends on $\boldsymbol{k}$ and the **mesh regularity**
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- Mild overestimation on reasonably fine meshes with Dirichlet conditions

- Analytical solution $u(x, y, t) = t^2 \sin(\pi x) \sin(\pi y)$
- $\Omega = [0; 1]^2$
- Fixed number of iterations (1,5,12)
- $\beta = 1.5\beta^\star$

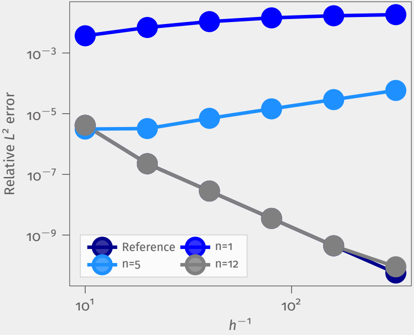
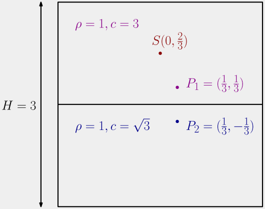
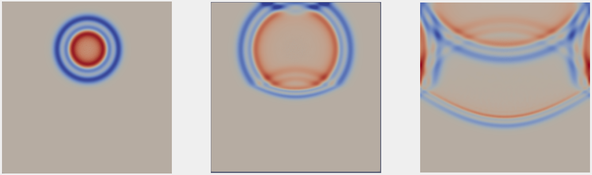


Figure. – HHO1 L^2-error : truncated splitting

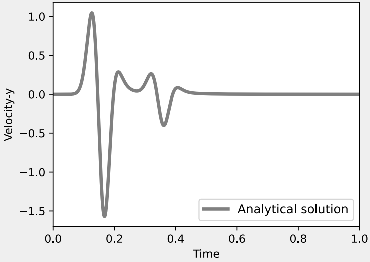
Wave propagation

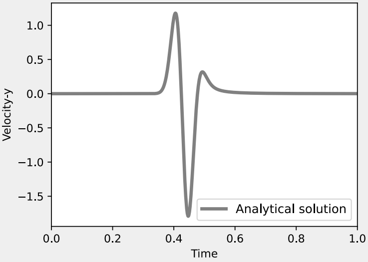
Wave propagation in heterogeneous domain with Dirac source at *S*.





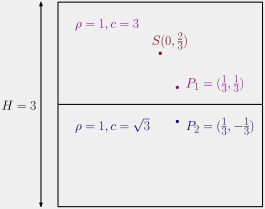
Analytical solution available until the reflections from the boundaries reached the sensor (Gar6more2D, Diaz & Abdelaaziz, open-source on Gitlab)

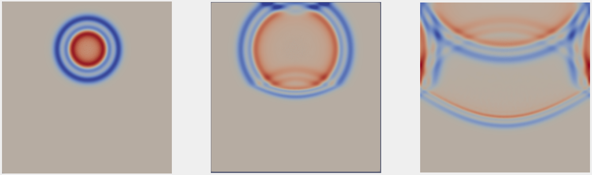




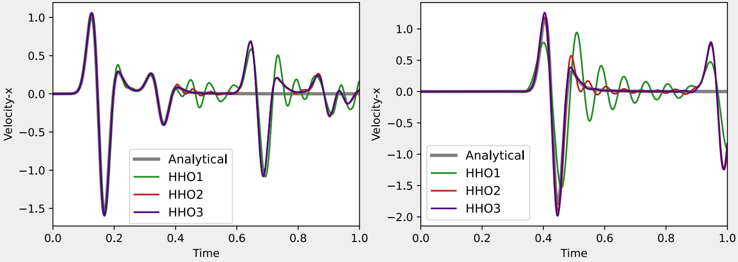
Wave propagation

Wave propagation in heterogeneous domain with Dirac source at S.



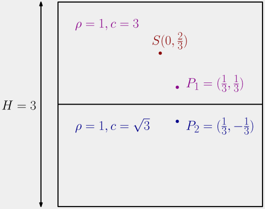


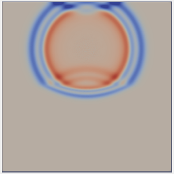
Measure of the velocity y-component on the same mesh for HHO{1,2,3} (P_1 left, P_2 right)

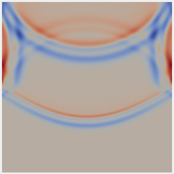


Wave propagation

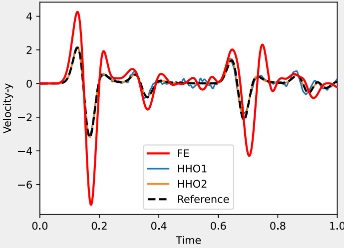
Wave propagation in heterogeneous domain with Dirac source at *S*.

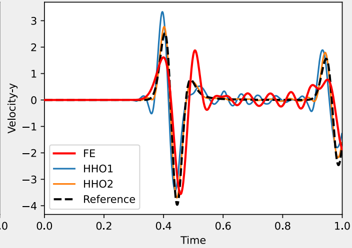






Measure of the velocity *y*-component for HHO and FE (same # DOFs ,P_1 left, P_2 right)





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NONLINEAR WAVE EQUATION

- Application to nonlinear wave equation,

$$\partial_t^2 u - \nabla \cdot (\alpha(\nabla u)\nabla u) = f$$

  with $\alpha(g) = (|g|^2 + 0.1)^{\frac{p-2}{2}}$, $p \in (1; \infty)$.
- Nonlinear stiffness term but linear stabilisation

$$\frac{1}{\Delta t^2}\begin{bmatrix} \mathbb{M}_{\mathcal{CC}} & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} U_{\mathcal{C}}^{n+1} - 2U_{\mathcal{C}}^{n} + U_{\mathcal{C}}^{n-1} \\ \cdot \end{pmatrix} + \begin{pmatrix} K_{\mathcal{C}}(U_{\mathcal{C}}^{n}, U_{\mathcal{F}}^{n}) \\ K_{\mathcal{F}}(U_{\mathcal{C}}^{n}, U_{\mathcal{F}}^{n}) \end{pmatrix} + \begin{bmatrix} \mathbb{S}_{\mathcal{CC}} & \mathbb{S}_{\mathcal{CF}} \\ \mathbb{S}_{\mathcal{FC}} & \mathbb{S}_{\mathcal{FF}} \end{bmatrix}\begin{pmatrix} U_{\mathcal{C}}^{n} \\ U_{\mathcal{F}}^{n} \end{pmatrix} = \begin{bmatrix} F_{\mathcal{C}}^{n} \\ 0 \end{bmatrix}$$

- Static coupling solved with Newton or with splitting :

$$\mathbb{S}_{\mathcal{FF}} U_{\mathcal{F}}^{n-1,m} = -K_{\mathcal{F}}(U_{\mathcal{C}}^{n}, U_{\mathcal{F}}^{n-1,m-1}) - \mathbb{S}_{\mathcal{FC}} U_{\mathcal{C}}^{n}$$

Experiment setting

Square domain $[0;1]^2$, Cartesian mesh, convergence condition on the splitting, 4 MPI processes on PC.
Measure value over time at a fixed point $(0.5, 0.5)$ and compare to the value on a more refined mesh (no analytical solution).

$$v_0(x,y) = \cos(\pi x)\cos(\pi y), u_0 = 0, f = 0$$

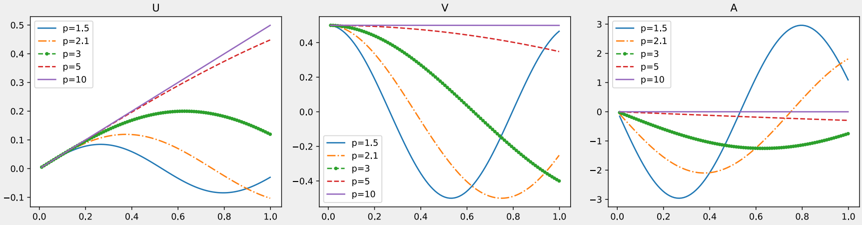

Figure. – Expected solutions, $p = 1.5, 2.1, 3, 5, 10$

Newton vs Splitting

- Measure value over time at $(0.5, 0.5)$ and compare to the value on a more refined mesh (no analytical solution) : error less than 0.1% for $\beta \in [10, 100]$

✓ More gain with larger β

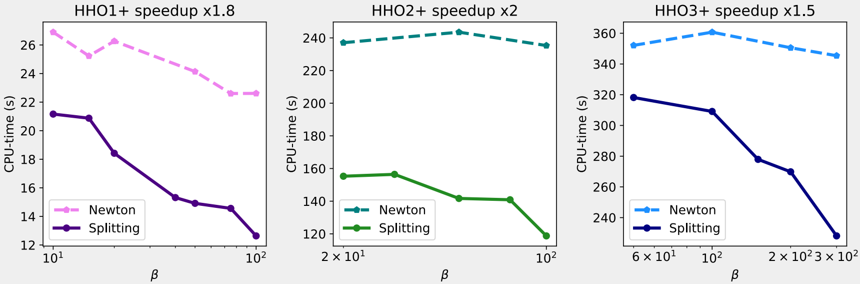

Figure. – computation times $p = 3$, $h = 0.03$, HHO1+,HHO2+, HHO3+

NEWTON VS SPLITTING

- Measure value over time at $(0.5, 0.5)$ and compare to the value on a more refined mesh (no analytical solution) : error less than 0.1% for $\beta \in [10, 100]$
- ✓ More gain with larger β
- ✓ More gain with larger meshes

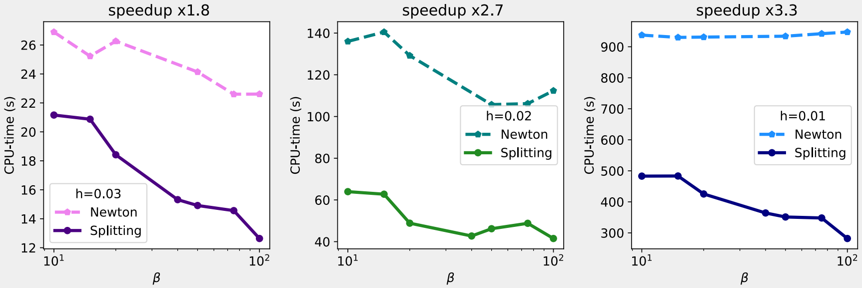

Figure. – HHO1+ : computation times $p = 3$, $h \in \{0.03, 0.02, 0.01\}$

Newton vs Splitting

- Measure value over time at $(0.5, 0.5)$ and compare to the value on a more refined mesh (no analytical solution) : error less than 0.1% for $\beta \in [10, 100]$

✓ More gain with larger β

✓ More gain with larger meshes

✓ More gain with larger p

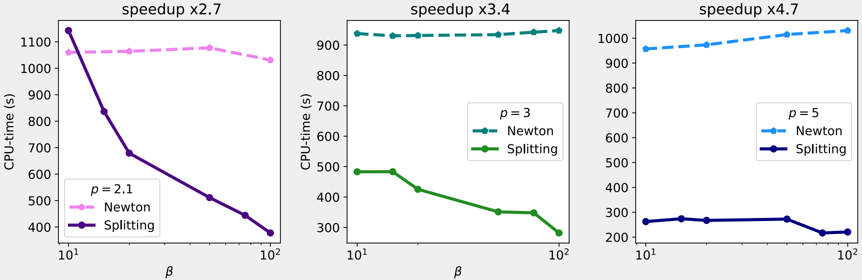

Figure. – HHO1+ : computation times $h = 0.01$, $p \in \{2.1, 3, 5\}$

- Measure value over time at $(0.5, 0.5)$ and compare to the value on a more refined mesh (no analytical solution) : error less than 0.1% for $\beta \in [10, 100]$
- ✓ More gain with larger β
- ✓ More gain with larger meshes
- ✓ More gain with larger p

Thank you for your attention